Non-perturbative phenomena in the three-dimensional random field Ising model

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Abstract

The systematic approach for the calculations of the non-perturbative contributions to the free energy in the ferromagnetic phase of the random field Ising model is developed. It is demonstrated that such contributions appear due to localized in space instanton-like excitations which exist only in dimensions $D \leq 3$. It is shown that away from the critical region such instanton solutions are described by the set of the mean-field saddle-point equations for the replica vector order parameter, and these equations can be formally reduced to the only saddle-point equation of the *pure system* in dimensions $(D-2)$. In the marginal case, $D = 3$, the corresponding non-analytic contribution is computed explicitly. Nature of the phase transition in the three-dimensional random field Ising model is discussed.

Key words:Quenched disorder, random fields, instantons, replica symmetry breaking.

1 Introduction

Despite extensive theoretical and experimental efforts during the past several decades very little is understood about the basic thermodynamic properties of the ferromagnetic Ising systems with quenched random fields (for reviews see e.g. [1]). In the most simple form the random field spin Ising systems can be described in terms of the Hamiltonian

$$H = -\sum_{<i,j>}^{N} \sigma_i \sigma_j - \sum_i h_i \sigma_i \tag{1}$$

where the Ising spins $\{\sigma_i = \pm 1\}$ are placed in the vertices of a D-dimensional lattice with the ferromagnetic interactions between the nearest neighbors, and quenched random fields $\{h_i\}$ are described by the symmetric Gaussian distribution:

$$P[h_i] = \left(2\pi h_0^2\right)^{-N/2} \exp\left[-\frac{1}{2h_0^2}\sum_{i=1}^{N} h_i^2\right] \tag{2}$$

where the parameter h_0 describes the strength of the random field

According to simple physical arguments by Imry and Ma [2] it should be expected that the dimensions D_c above which the ferromagnetic ground state is stable at low temperatures (it is called the lower critical dimension) must be $D_c = 2$ (unlike the pure Ising systems where $D_c = 1$) Indeed, if we try to test the stability of the ferromagnetic state by flipping the sign of the magnetisation in a large region of linear size L, we will find two competing effects: a possible gain of energy, due go alignment with the random magnetic field, which scales as $E_h = h_0 L^{D/2}$; and the loss of energy, due to the creation of an interface, which scales as L^{D-1}. These estimates show that below dimensions $D_c = 2$ for any non-zero value of the field h_0 at sufficiently large sizes L the two energies are getting comparable, and therefore no spontaneous magnetization should be present. On the other hand, at dimensions greater than $D_c = 2$, the energy of the interface is always bigger than E_h. Therefore these excitations will not destroy the long range order, and a ferromagnetic transition should be present. These naive (but physically correct) arguments was later confirmed by a rigorous proof by Imbrie [3].

On the other hand, a perturbative study of the phase transition shows that, as far as the leading large scale divergences are concerned, the strange phenomenon of a dimensional reduction is present, such that the critical exponents of the system in the dimension D appear to be the same as those of the ferromagnetic system without random fields in the dimension d=D-2 [4]. Since the lower critical dimension of the pure

Ising model is equal to 1, this result would imply that the lower critical dimension of the random field Ising model must be equal to 3, in contradiction with the rigorous results. Actually, the procedure of summation of the leading large scale divergences could give the correct result only if the Hamiltonian in the presence of random fields has only one minimum. In this case the dimensional reduction can be rigorously shown to be exact, by the use of supersymmetric arguments [5].

However, one can easily see that as soon as the temperature is close enough to the putative critical point (as well as in the whole low temperature region), there are local values of the magnetic fields for which the free energy has more than one minimum [6]. In this situation there is no reason to believe that the supersymmetric approach should give the correct results and therefore the dimensional reduction is not grounded. Thus, the above arguments settle the controversy about the lower critical dimension of the random field Ising model in favour of the value $D_c = 2$.

Although at present we understand that at $D > 2$ the low temperature state of the RFIM must be ferromagnetic, the nature of the phase transition from the paramagnetic to the ferromagnetic state is still a mystery. The only well established fact here is that the upper critical dimensionality (the dimensionality above which the critical phenomena are described by the mean-field critical exponents) of such systems is equal to 6 (unlike pure systems, where it is equal to 4). What is going on in the close vicinity of the phase transition, at dimensions $D < 6$ is not known. And it is not just the question of what are the true values of the critical exponents. Even the nature of the phase transition (is it of the second or of the first order) is nor clear. Moreover, at least in some cases there are indications in favour of the existence of the intermediate spin-glass state (with broken replica symmetry) separating paramagnetic and ferromagnetic phases [7, 8].

The key problem here is the effect produced by the non-perturbative contributions which are coming from numerous local minima states. In the present study, considering these contributions in the low temperature ferromagnetic phase (away from the critical point) at the *finite* strength of random fields h_0, I would like to "rehabilitate" the marginal character of the dimensionality $D = 3$ as well as of the trick of dimensional reduction $D \to (D - 2)$. The point is that although numerous local minima configurations always exist in the ferromagnetic phase of the RFIM, they are not always statistically relevant for the thermodynamic properties of the macroscopic system. In this paper I am going to demonstrate, first in terms of simple heuristic arguments (Section 2), and then using more rigorous replica technique (Section 3), that the non-perturbative states yield (non-analytic) contribution to the thermodynamics only in dimensions $D \leq 3$. In terms of the qualitative heuristic arguments this contribution appears due to rare large cluster spin flips. In terms of the non-perturbative replica formalism [9, 10] such configurations are described by the instanton type solutions of the mean-field saddle-point equations for the replica vector order parameter with broken replica symmetry. These equations can be formally reduced to the only saddle-point equation of the corresponding *pure system* in dimensions $(D - 2)$ (this fact has been noted first in [11]), and then one can easily note that localized instanton-like solutions formally exist only in dimensions $D < 3$. In this sense the dimension $D = 3$ is marginal: in dimensions $D > 3$ the non-perturbative states are irrelevant, while in dimensions $D < 3$ they yield finite non-analytic contribution (finally, when approaching the dimension $D = 2$ from above these states become so much relevant that they destroy the ferromagnetic ground state of the system). Similar instanton-like configurations in the presence of the homogeneous external magnetic field have been recently studied in Ref.[12]. Although formally the saddle-point equation in dimension $D = 3$ (in the zero external field) have no localized instanton solutions, nevertheless, as it often happens in the marginal situations, the the relevant contributions can be taken into account in terms of the instanton-like configurations containing the *soft-mode parameter* (which is the size of the instanton).

Away from the critical point the concentration of the instantons (or the flipping clusters) is exponentially small, so that they can be treated as independent finite energy excitations. However, when approaching the putative phase transition point (from below) the typical distance between instantons eventually becomes comparable with their size. In this situation the present scheme of calculations breaks down, as the state of the system becomes just a mess of locally ordered "up" and "down" regions, when it is impossible to separate the degrees of freedom connected with the flipped clusters from those of the ferromagnetic background. It is shown however, that at the *finite* strength of the random fields this happens at temperatures where the system can still be described at the mean-field level, and it is argued that in this situation it would be reasonable to expect the phase transition of the *first order* into the disordered state (Section 4).

2 Heuristic arguments

2.1 Perturbative contributions

To compare the perturbative and the non-perturbative effects in the random field Ising model let us consider its Ginzburg-Landau continuous representation:

$$H[\phi(\mathbf{x})] = \int d^D x \left[\frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}\tau\phi^2(\mathbf{x}) + \frac{1}{4}g\phi^4(\mathbf{x}) - h(\mathbf{x})\phi(\mathbf{x})\right] \tag{3}$$

Here $\tau = (T/T_c - 1)$ ($|\tau| \ll 1$) is the reduced temperature parameter (for simplicity in what follows it will be supposed that $T_c = 1$) and the random fields $h(\mathbf{x})$ are described by the Gaussian distribution,

$$P[h(\mathbf{x})] = p_0 \exp\left(-\frac{1}{2h_0^2}\int d^D x\, h^2(\mathbf{x})\right), \tag{4}$$

where h_0 is the parameter which describes the strength of the random field, and p_0 is an irrelevant normalization constant. The system will be considered in the low-temperature ferromagnetic phase, so that the reduced temperature parameter τ will be taken to be negative, $\tau = -|\tau|$. To neglect the effects of the thermal fluctuations (away from the critical point), it will be assumed that the absolute value of the parameter $|\tau|$ is not too small:

$$|\tau| \gg g^{2/(4-D)} \equiv \tau_{GL} \tag{5}$$

which is the usual Ginzburg-Landau condition. The only relevant spatial scale in the system described by the Hamiltonian (3) is the correlation length:

$$R_c(\tau) \sim |\tau|^{-1/2} \tag{6}$$

(which under condition (5) is described by the mean-field critical exponent $\nu = 1/2$). The statistical properties of the system at scales bigger than the correlation length can be studied in terms of the saddle-point configurations defined by the equation

$$-\Delta\phi(\mathbf{x}) - |\tau|\phi(\mathbf{x}) + g\phi^3(\mathbf{x}) = h(\mathbf{x}) \tag{7}$$

In the absence of the random fields the ferromagnetic ground state of the system is described by the homogeneous configuration $\phi_0 = \sqrt{|\tau|/g}$, and it has the free energy density $f_0 = -\tau^2/(4g)$. In the usual perturbative approach the effects produced by the random fields can be taken into account e.g. in the following way. After rescaling the fields

$$\phi(\mathbf{x}) = \left(\frac{|\tau|}{g}\right)^{1/2}\tilde{\phi}(\mathbf{x}/R_c), \tag{8}$$

instead of eq.(3) one obtains the system which is described by the rescaled Hamiltonian

$$H\left[\tilde{\phi}(\mathbf{z})\right] = \frac{|\tau|^{\frac{(4-D)}{2}}}{g}\int d^D z\left[\frac{1}{2}\left(\nabla\tilde{\phi}(\mathbf{z})\right)^2 - \frac{1}{2}\tilde{\phi}^2(\mathbf{z}) + \frac{1}{4}\tilde{\phi}^4(\mathbf{z})\right] - \frac{|\tau|^{\frac{(1-D)}{2}}}{\sqrt{g}}\int d^D z\, \tilde{h}(\mathbf{z})\tilde{\phi}(\mathbf{z}) \tag{9}$$

where $\mathbf{z} \equiv \mathbf{x}/R_c = |\tau|^{1/2}\mathbf{x}$. Here rescaled random fields

$$\tilde{h}(\mathbf{x}/R_c) = R_c^{-D}\int_{|\mathbf{x}'-\mathbf{x}|<R_c} d^D x'\, h(\mathbf{x}') \tag{10}$$

are described by the distribution function

$$P[\tilde{h}(\mathbf{z})] = \tilde{p}_0 \exp\left(-\frac{1}{2h_0^2|\tau|^{D/2}}\int d^D z \tilde{h}^2(\mathbf{z})\right), \tag{11}$$

which is characterized by the mean square value

$$\overline{\tilde{h}^2} = |\tau|^{D/2} h_0^2 \tag{12}$$

The ground state configurations of the fields $\tilde{\phi}(\mathbf{z})$ are now defined by the saddle-point equation

$$-\Delta\tilde{\phi}(\mathbf{z}) - \tilde{\phi}(\mathbf{z}) + \tilde{\phi}^3(\mathbf{z}) = \frac{g^{1/2}}{|\tau|^{3/2}}\tilde{h}(\mathbf{z}) \tag{13}$$

In the absence of the random fields ($\tilde{h} \equiv 0$) the above equation would yield the trivial homogeneous ferromagnetic ground state solution $\tilde{\phi}(\mathbf{z}) = 1$ (or $\tilde{\phi}(\mathbf{z}) = -1$). In the presence of the random field term the ground state solution can be represented as

$$\tilde{\phi}_{gs}(\mathbf{z}) = 1 + \psi(\mathbf{z}) \tag{14}$$

where $\psi(\mathbf{z})$ describes small ($\psi \ll 1$) spatial fluctuations of the ground state configuration. Solving eq.(13) in the *perturbative* way (considering its r.h.s term as the small perturbation), for the typical value of these spatial fluctuations one easily finds:

$$\overline{\psi^2} \sim \frac{g h_0^2}{|\tau|^{(6-D)/2}} + O\left(\left(\frac{g h_0^2}{|\tau|^{(6-D)/2}}\right)^2\right) \tag{15}$$

Substituting the solution eq.(14) into the Hamiltonian eq.(9) for the free energy density of the ground state configuration one gets:

$$f_{gs} = \frac{1}{V} H\left[\tilde{\phi}_{gs}\right] = -\frac{|\tau|^{\frac{(4-D)}{2}}}{4g}\left(1 + \delta\tilde{f}\right)\frac{1}{R_c^D} = -\frac{|\tau|^2}{4g}\left(1 + \delta f\right) \tag{16}$$

where δf is the random quantity with the typical value

$$\overline{(\delta f)^2} \sim \frac{g h_0^2}{|\tau|^{(6-D)/2}} + O\left(\left(\frac{g h_0^2}{|\tau|^{(6-D)/2}}\right)^2\right) \tag{17}$$

We see that the perturbation expansion goes in powers of the (small) parameter

$$\frac{g h_0^2}{|\tau|^{(6-D)/2}} \ll 1 \tag{18}$$

which means that such perturbative approach is bounded by the condition

$$|\tau| \gg \tau_*(h_0) = \left(g h_0^2\right)^{\frac{2}{6-D}} \tag{19}$$

On the other hand, according to eq.(9), the typical value of the *thermal* fluctuations

$$\langle\tilde{\phi}^2\rangle \sim g|\tau|^{-\frac{4-D}{2}} \tag{20}$$

In the pure system we can neglect the effects of the thermal fluctuations provided $\langle\tilde{\phi}^2\rangle \ll 1$, which imposes the usual Ginzburg-Landau condition, eq.(5). Here, in the presence of random fields, the thermal fluctuations can be neglected provided they are small not only compared to the average value of the ferromagnetic order parameter (which in the present notations is equal to one), but also compared to the typical value of its random spatial variations (which is described by the field ψ, eq.(14)):

$$\langle\tilde{\phi}^2\rangle \ll \overline{\psi^2} \tag{21}$$

Substituting here eqs.(15) and (20) we find one more bound for the temperature parameter $|\tau|$:

$$|\tau| \ll h_0^2 \tag{22}$$

Thus, taking into account eq.(19), we conclude that the perturbative mean-field consideration of the present system is legitimate in the temperature interval

$$\left(g h_0^2\right)^{\frac{2}{6-D}} \ll |\tau| \ll h_0^2 \tag{23}$$

which automatically requires that the typical value of the random field can not be too small:

$$h_0^2 \gg g^{\frac{2}{4-D}} \tag{24}$$

Note that under this condition

$$\tau_{GL} \ll \tau_*(h_0) \tag{25}$$

Another words, when approaching the phase transition from below (which means reducing the absolute value of the temperature parameter $|\tau|$), first one arrives to the crossover temperature $\tau_*(h_0)$ when the random fields can not be considered as small perturbations any more (while the thermal fluctuations can still be neglected), and only after that, at $|\tau| \sim \tau_{GL}$ (where the perturbation theory is no more valid) the thermal fluctuations would become important.

In the particular case of the dimension $D = 3$ (which is the main focus of our further study) the expansion parameter of the perturbation theory is

$$\frac{gh_0^2}{|\tau|^{3/2}} \ll 1 \tag{26}$$

The crossover temperatures are

$$\begin{aligned} \tau_{GL} &= g^2 \\ \tau(h_0) &= \left(gh_0^2\right)^{2/3} \end{aligned} \tag{27}$$

and the consideration is limited by the temperature interval

$$\tau(h_0) \ll |\tau| \ll h_0^2 \tag{28}$$

which can exist provided the typical value of the random fields is not too small:

$$h_0 \gg g \tag{29}$$

Within these bounds the perturbative calculation of the free energy can be expressed in the form of eqs.(16)-(17).

2.2 Non-perturbative contributions

Besides perturbative contributions described above, one can observe completely different type of thermal excitations. Let us suppose that in the low temperature phase the system has the global ferromagnetic magnetization directed "up", and let us consider a spatial island with sufficiently large linear size L, in which the average value of the (random) field $\tilde{h}$ is negative and sufficiently strong. Then one can easily note that in addition to the state "up" (with slightly modified value of the order parameter), another local minimum with orientation "down" can be formed within this island. Considering our system in terms of the rescaled Hamiltonian, eq.(9), one can easily see that this alternative state can exist only if the gain in the energy due to the interaction with the random field,

$$E_h \sim -\frac{|\tau|^{\frac{(1-D)}{2}}}{\sqrt{g}} L^D |\tilde{h}| \tag{30}$$

overruns the loss of energy due to the creation of the interface,

$$E_f \sim +\frac{|\tau|^{\frac{(4-D)}{2}}}{g} L^{D-1} \tag{31}$$

These estimates demonstrate that for a given value of L such double-state situation in the considered island is created provided

$$|\tilde{h}| > h_c(L) \sim \frac{|\tau|^{3/2}}{\sqrt{g}L} \tag{32}$$

which has a finite probability for any finite value of L.

Let us consider the statistical properties of such type of excitations in more detail. In terms of the Hamiltonian, eq.(9), the energy of the spherical region of the radius L in which the magnetization is flipped "down" can be represented as the sum of two contributions:

$$E(L) \;=\; (const)\frac{|\tau|^{\frac{(4-D)}{2}}}{g} L^{D-1} - V(L) \tag{33}$$

The first term here is the positive energy due to creation of the spherical interface, while the second contribution is the random energy which appear due to the interaction with the random field, which in the continuous limit it can be represented as

$$V(L) \;=\; 2\frac{|\tau|^{\frac{(1-D)}{2}}}{\sqrt{g}} \int_{|\mathbf{z}|<L} d^D z \, \tilde{h}(\mathbf{z}) \tag{34}$$

According to this definition,

$$\overline{V^2(L)} \;\sim\; \frac{|\tau|^{(1-D)}}{g} L^D \, \overline{\tilde{h}^2} \;\sim\; \frac{|\tau|^{\frac{(2-D)}{2}}}{g} L^D \, h_0^2 \tag{35}$$

Since the typical value of the random energy $\left[\overline{V^2(L)}\right]^{1/2}$ scales as $L^{D/2}$ we conclude that in dimensions $D > 2$ the first term in the total energy $E(L)$, eq.(33), must dominate at large scales, and therefore $E(L)$ is (on average) a growing function of L (this is nothing else but the familiar Imry-Ma arguments [2] which explain why the ferromagnetic state is stable at $D > 2$). The point, however, is that $E(L)$ is the *random* function, and it grows with L only *on average*, while for a given realization of disorder it can have local minima at various (large) values of L.

The probability distribution which describes the statistics of the random functions $V(L)$ is given by

$$\mathcal{P}[V(L)] \;=\; \int \mathcal{D}\tilde{h}(\mathbf{z}) \;\; P[\tilde{h}(\mathbf{z})] \prod_L \left[\delta\left(\frac{2|\tau|^{\frac{(1-D)}{2}}}{\sqrt{g}} \int_{|\mathbf{z}|<L} d^D z \, \tilde{h}(\mathbf{z}) - V(L) \right) \right] \tag{36}$$

where $P[\tilde{h}(\mathbf{z})]$ is the probability distribution function, eq.(11). Performing straightforward Gaussian integrations one obtains

$$\mathcal{P}[V(L)] \;=\; (const) \; \exp\left[-\frac{g}{8h_0^2|\tau|^{\frac{(2-D)}{2}} S_D} \int_1^\infty \frac{dL}{L^{D-1}} \left(\frac{dV(L)}{dL} \right)^2 \right] \tag{37}$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the square of the unite sphere in D dimension. Since we are going to consider only large scales, the above result is given in the continuous limit representation containing the ultraviolet cut-off of the order of the correlation length, which in the present notations is equal to one. We see, that in accordance with the physical meaning of the function $V(L)$, its statistical distribution depends only on its derivative (the constant term in this function is irrelevant), and therefore the problem would become well defined only if we fix a value of this function at a given L. For simplicity, let us assume that at the lowest possible scale $V(L = 1) = 0$.

The above probability distribution function, eq.(37), can be used to estimate the probability that the random function $E(L)$ has at least one local minimum at scales larger than a given scale L. Since the value of the energy $E(L)$ in a putative minimum growth with L, its probability must be small at large scales. In this situation the sufficient condition for the existence of a minimum somewhere beyond a given size L is $\frac{dE(L)}{dL} < 0$, or

$$\frac{dV(L)}{dL} > (const)\frac{|\tau|^{\frac{(4-D)}{2}}}{g} L^{D-2} \tag{38}$$

The probability $P_{min}(L)$ that the above condition is satisfied at a unit length at the given size L can be easily estimated using the general distribution function (37). Formally it can be obtained by integrating $\mathcal{P}[V(L)]$ over all functions $V(L)$ conditioned by eq.(38). It is clear, however, that with the exponential accuracy, the result of such integration is defined by the lower bound for the derivative $dV(L)/dL$. Therefore, substituting (at a given value of L) $\frac{dV(L)}{dL} = (const)\frac{1}{g}|\tau|^{\frac{(4-D)}{2}} L^{D-2}$ into eq.(37), with the exponential accuracy one gets:

$$P_{min}(L) \sim \exp\left[-(const)\frac{|\tau|^{\frac{(6-D)}{2}}}{gh_0^2}L^{D-3}\right] \tag{39}$$

Thus we have derived very important property of the random function $E(L)$, eq.(33): although, according to eq.(35) this function (in dimensions $D > 2$) on average grows with L, the probability of finding a local minimum of this function in dimensions $D < 3$, according to eq.(39), also grows with L. The situation in the three-dimensional case is marginal, and it is not quite clear to what extent the above simple arguments are grounded for $D = 3$ (as usual in the marginal situations, more rigorous methods has to be used). Nevertheless, if we formally substitute $D = 3$ into eq.(39), we would have to conclude that the probability of a local minimum becomes independent of the size of the flipped cluster. Since the value of $P_{min}(L)|_{(D=3)}$ is exponentially small (in the parameter $|\tau|^{3/2}/(gh_0^2) \gg 1$, eq.(26)), the contribution to the free energy of such type of cluster excitations with the exponential accuracy is defined by their probabilities (while their energies define a pre-exponential factor).Thus we can estimate the non-perturbative part of the free energy density in the ferromagnetic phase of the 3D random-field Ising model as

$$\Delta f \ \sim \ \exp\left(-(const)\frac{|\tau|^{3/2}}{gh_0^2}\right) \tag{40}$$

In the next section we shall re-derive this result in terms of the systematic replica approach, which, in particular, allows to calculate the $(const)$ factor.

3 Non-perturbative replica approach

The general scheme of the non-perturbative replica calculations has already been discussed in detail in the recent papers [9, 10]. Here we repeat it just in brief. Let us consider a general D-dimensional random system described by a Hamiltonian $H[\phi(\mathbf{x}); h(\mathbf{x})]$, where $\phi(\mathbf{x})$ is a field which defines the microscopic state of the system, and $h(\mathbf{x})$ are quenched random parameters. Let us suppose that in addition to the ground state, this system has another thermodynamically relevant local minima states located "far away" from the ground state and separated from it by a finite barrier of the free energy. In other words, it is *supposed* that the partition function (of a given sample) can be represented in the form of two separate contributions:

$$Z = \int \mathcal{D}\phi(\mathbf{x})\, e^{-\beta H} \ = \ e^{-\beta F_0} + e^{-\beta F_1} \ \equiv \ Z_0 \ + \ Z_1 \tag{41}$$

where F_0 is the contribution coming from the vicinity of the ground state, and F_1 is the contribution of the local minima states. Then, for the averaged over disorder total free energy we find:

$$\mathcal{F} = -\frac{1}{\beta}\overline{\ln Z} = \overline{F_0} - \frac{1}{\beta}\overline{\ln\left[1 + Z_1 Z_0^{-1}\right]} \tag{42}$$

The second term in the above equation, which will be denoted by ΔF, can be represented in the form of the series:

$$\Delta F = -\frac{1}{\beta}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\overline{Z_1^m Z_0^{-m}} = -\frac{1}{\beta}\lim_{n\to 0}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}Z_n(m) \tag{43}$$

where

$$Z_n(m) = \prod_{b=1}^{m}\int \mathcal{D}\phi_b^{(1)} \prod_{c=1}^{n-m}\int \mathcal{D}\phi_c^{(0)}\ e^{-\beta H_n\left[\phi_1^{(1)},...,\phi_m^{(1)},\phi_1^{(0)},...,\phi_{n-m}^{(0)}\right]} \tag{44}$$

is the replica partition function ($H_n[\boldsymbol{\phi}]$ is the corresponding replica Hamiltonian), in which the replica symmetry in the n-component vector field ϕ_a $(a = 1, ..., n)$ is assumed to be broken. Namely, it is supposed that the saddle-point equations

$$\frac{\delta H_n[\boldsymbol{\phi}]}{\delta \phi_a(\mathbf{x})} \ = \ 0\ , \quad (a = 1, ..., n) \tag{45}$$

have non-trivial solutions with the RSB structure

$$\phi_a^*(\mathbf{x}) = \begin{cases} \phi_1(\mathbf{x}) & \text{for } a = 1, ..., m \\ \\ \phi_0(\mathbf{x}) & \text{for } a = m+1, ..., n \end{cases} \tag{46}$$

with $\phi_1(\mathbf{x}) \neq \phi_0(\mathbf{x})$, so that the integration in the above partition function, eq.(44), goes over fluctuations in the vicinity of these components. It should be stressed that to be thermodynamically relevant, the RSB saddle-point solution, eq.(46) should be *localized* in space and characterized by a *finite* space size $R(m)$ and finite energies $E(m) = H_n[\boldsymbol{\phi}^*]$. In this case the partition function, eq.(44), will be proportional to the entropy factor $V/R^D(m)$ (where V is the volume of the system), and the corresponding free energy contribution ΔF, eq.(43), will be extensive quantity.

Thus, to compute the non-perturbative free energy contribution one should find all saddle-point RSB solutions $\phi_a^*(\mathbf{x})$, eq.(45) and their corresponding energies $E(m)$ (in the limit $n \to 0$), and finally one has to sum up the series

$$\Delta F = -\frac{V}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} R^{-D}(m) \left(\beta \det \hat{T}\right)_{n=0}^{-1/2} e^{-\beta E(m)} \tag{47}$$

Here

$$T_{aa'} = \frac{\delta^2 H[\boldsymbol{\phi}]}{\delta\phi_a \delta\phi_{a'}}\Big|_{\boldsymbol{\phi}=\boldsymbol{\phi}^*} \tag{48}$$

and the Hessian pre-exponential factor $(\det \hat{T})^{-1/2}$ (taken in the limit $n \to 0$) appears due to the integration over the replica fluctuations in the vicinity of the RSB solutions, eq.(46). Note that in the present approach the procedure of analytic continuation $n \to 0$ is quite similar to that in the usual replica theory [14]: whenever the parameter n becomes an algebraic factor (and not the summation parameter, or the matrix size, etc.), it can safely be set to zero right away.

Coming back to the original Hamiltonian, eq.(3), and following the standard procedure, after the Gaussian averaging of the replicated partition function over the random fields $h(\mathbf{x})$, one obtains the replica Hamiltonian

$$H_n[\boldsymbol{\phi}] = \int d^D x \left[\frac{1}{2}\sum_{a=1}^{n} (\nabla\phi_a)^2 - \frac{1}{2}|\tau| \sum_{a=1}^{n} \phi_a^2 + \frac{1}{4} g \sum_{a=1}^{n} \phi_a^4 - \frac{1}{2} h_0^2 \sum_{a,b=1}^{n} \phi_a \phi_b\right] \tag{49}$$

The saddle-point configurations of the fields $\phi_a(x)$ are defined by the equations

$$-\Delta\phi_a(\mathbf{x}) - |\tau|\phi_a(\mathbf{x}) + g\phi_a^3(\mathbf{x}) - h_0^2 \left(\sum_{b=1}^{n} \phi_b(\mathbf{x})\right) = 0 \tag{50}$$

Below it will be demonstrated that besides the obvious (replica symmetric) ferromagnetic solution $\phi_a(\mathbf{x}) = \phi_0 = \sqrt{|\tau|/g}$ these equations have non-trivial localized in space instanton-like solutions with the RSB structure:

$$\phi_a^*(\mathbf{x}) = \begin{cases} \sqrt{\frac{|\tau|}{g}}\, \psi_1(\mathbf{x}\sqrt{|\tau|}) & \text{for } a = 1, ..., m \\ \\ \sqrt{\frac{|\tau|}{g}}\, \psi_0(\mathbf{x}\sqrt{|\tau|}) & \text{for } a = m+1, ..., n \end{cases} \tag{51}$$

These solutions are characterised by two non-trivial functions $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z})$ (where $\mathbf{z} = \mathbf{x}\sqrt{|\tau|}$). Substituting these rescaled fields into the saddle-point eqs(50) and into the Hamiltonian, eq.(49), we find that (in the limit $n \to 0$) the instanton configuration $\{\psi_1(\mathbf{z}), \psi_0(\mathbf{z})\}$ is defined by the equations

$$\begin{aligned} -\Delta\psi_1 - \psi_1 + \psi_1^3 - \lambda(m)(\psi_1 - \psi_0) &= 0 \\ -\Delta\psi_0 - \psi_0 + \psi_0^3 - \lambda(m)(\psi_1 - \psi_0) &= 0 \end{aligned} \tag{52}$$

and its energy is

$$E(m) = m\frac{|\tau|^{2-D/2}}{g} \int d^D z \left[\frac{1}{2} \left[(\nabla\psi_1)^2 - (\nabla\psi_0)^2 \right] - \frac{1}{2} \left[\psi_1^2 - \psi_0^2 \right] + \frac{1}{4} \left[\psi_1^4 - \psi_0^4 \right] - \frac{1}{2}\lambda(m) \left[\psi_1 - \psi_0 \right]^2 \right] \quad (53)$$

where

$$\lambda(m) \;=\; \frac{h_0^2 m}{|\tau|} \quad (54)$$

We are looking for the localized in space (spherically symmetric) solutions of the eqs.(52), such that the two functions $\psi_1(r)$ and $\psi_0(r)$ (where $r = |\mathbf{z}|$) are different from each other in a finite region of space, and at large distances they both sufficiently quickly approach the same value $\psi_0 = 1$, so that the integral in eq.(53) will be converging.

Keeping in mind the qualitative arguments of the previous Section it will be assumed that the parameters of the model satisfy the requirements, eqs.(23)-(25). In this case the considered theory, eqs.(52)-(54), contains the large parameter $\lambda(m) \gg 1$ (for any $m = 1, 2, ...$), so that, according to eqs.(52), the two fields ψ_1 and ψ_0 must be close to each other. Redefining,

$$\begin{aligned} \psi_1(r) &= \psi(r) + \frac{1}{\lambda}\chi(r) \\ \psi_0(r) &= \psi(r) - \frac{1}{\lambda}\chi(r) \end{aligned} \quad (55)$$

in the leading order in $\lambda^{-1} \ll 1$ instead of eqs.(52) we get much more simple equations:

$$\begin{aligned} -\Delta\psi - \psi + \psi^3 - 2\chi &= 0 \\ -\Delta\chi + (3\psi^2 - 1)\chi &= 0 \end{aligned} \quad (56)$$

which *contain no parameters*. For the energy of the configurations described by the two fields $\psi(r)$ and $\chi(r)$ instead of eq.(53) (again, in the leading order in λ^{-1}) we find the value, which does not depend on the summation parameter m,

$$E \;=\; \frac{|\tau|^{\frac{6-D}{2}}}{h_0^2 g}\, E_0 \quad (57)$$

where

$$E_0 \;=\; \int d^D z \left[(\nabla\psi)(\nabla\chi) + (\psi^3 - \psi)\chi - \chi^2 \right] \quad (58)$$

is the universal quantity which depends only on the dimensionality of the system. Considering only spherically symmetric configurations, eqs.(56) can be represented as

$$\begin{aligned} -\frac{d^2\psi}{dr^2} - \frac{D-1}{r}\frac{d\psi}{dr} + \psi^3 - 2\chi &= 0 \\ -\frac{d^2\chi}{dr^2} - \frac{D-1}{r}\frac{d\chi}{dr} + (3\psi^2 - 1)\chi &= 0 \end{aligned} \quad (59)$$

Now as a matter of a simple algebraic exercise one can easily check that taking

$$\chi \;=\; -\frac{1}{r}\frac{d\psi}{dr} \quad (60)$$

the above two equations (59) can be reduced to one equation

$$-\frac{d^2\psi}{dr^2} - \frac{D-3}{r}\frac{d\psi}{dr} - \psi + \psi^3 = 0 \tag{61}$$

for the only function $\psi(r)$. The corresponding energy, eq.(58), of the configurations described by eqs.(60) and (61) can be reduced to

$$E_0 \;=\; (D-2)S_D \int dr\; r^{D-3} \left[\frac{1}{2}\left(\frac{d\psi}{dr}\right)^2 \;+\; \frac{1}{4}(\psi^2-1)^2\right] \tag{62}$$

Thus, we have reduced the problem of the non-perturbative excitations in the D-dimensional random field Ising model to the study of similar instanton-like saddle-point configurations in the corresponding *pure* system in dimensions $(D-2)$.

Physically relevant solutions of the saddle-point equation (61) must be such that in the limit $r \to \infty$ the function $\psi(r)$ sufficiently quickly approaches the value $\psi_0 = 1$ (so that the integral in eq.(62) would be converging). It is evident, that at $D > 3$ (which corresponds to the "effective" dimension $D_{eff} \equiv (D-2) > 1$) eq.(61) has no such solutions. The formal solutions of this type and their contributions to the free energy in dimensions $2 < D < 3$ (which correspond to $0 < D_{eff} < 1$) has been first considered in Ref.[11], and then studied in detail in Ref.[10].

Here we are going to concentrate on the marginal case $D = 3$, when eq.(61) reduces to

$$\frac{d^2\psi}{dr^2} \;=\; \psi^3 \;-\; \psi \tag{63}$$

(where $0 \le r < +\infty$). Although this equation has no instanton solutions, the configurations of the type

$$\psi(r) \;=\; \tanh\left(\frac{r-L}{\sqrt{2}}\right) \tag{64}$$

(which are the formal solutions of eq.(63) for $-\infty < r < +\infty$) at large enough values of the parameter L can be considered as the soft-mode "quasi-instantons". Using eq.(62) one can easily find that at $L \gg 1$ the energy of such configuration is weakly dependent on the soft-mode parameter L:

$$E_0(L) \;\simeq\; E_* \;-\; 8\sqrt{2}\pi \exp(-4L/\sqrt{2}) \tag{65}$$

where

$$E_* \;=\; \frac{8\sqrt{2}}{3}\pi \tag{66}$$

is the energy of the infinite-size configuration.

It is clear that the energy $E_0(L)$ monotonously decreases with L, so that the true minimum is achieved when the above quasi-instanton configuration completely disappears. Therefore the configurations described by eq.(64) represents continuous spectrum of thermal excitations described by one parameter L. Correspondingly their contribution to the free energy is given by the summation (the integration) over all possible sizes L. In terms of the original spatial notations ($\mathbf{x} = \mathbf{z}|\tau|^{-1/2} \equiv \mathbf{z}R_c$) the size of the above quasi-instanton is

$$R \;=\; L\, R_c \tag{67}$$

Thus, coming back to the general expression for the off-perturbative part of the free energy (with $D = 3$ and $\beta = 1$) eq.(47), (where the energy $E(m)$ and the size R are defined in eqs.(57) and (67)), and introducing here the integration over the soft-mode parameter L, we obtain

$$\Delta F = -\frac{1}{\beta}\int_1^\infty dL\, \frac{V}{(LR_c)^3} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\; \exp\left(-\frac{|\tau|^{3/2}}{h_0^2 g} E_0(L)\right) \tag{68}$$

(here the ratio $V/(LR_c)^3$ is the entropy factor which yields the number of positions of the instanton with the linear size (LR_c) in the three-dimensional volume V). Note that for the derivation of ΔF with the exponential accuracy, in the considered range of parameters, eqs.(23)-(25), the contribution of the fluctuations (contained in the term $(\det \hat{T})^{-1/2}_{n=0}$, eq.(48)) can be neglected [10]. Substituting here eq.(65), and neglecting all pre-exponential contributions, for the density of the non-perturbative part of the free energy we finally get (cf. eq(40))

$$\Delta f \ \sim \ \exp\left(-E_* \frac{|\tau|^{3/2}}{g h_0^2}\right) \tag{69}$$

where $E_* = \frac{8\sqrt{2}}{3}\pi$. Note that the validity of this result is limited by the condition: $(gh_0^2)^{2/3} \ll |\tau| \ll h_0^2$.

4 Discussion

The present study of the non-perturbative phenomena in the random field Ising model has been done in terms of the mean-field local minima configurations in the low-temperature ferromagnetic state. The approach completely neglects thermal fluctuations, which is justified, on one hand, provided the temperature is not too close to the putative phase transition point, and on the other hand, provided the temperature is not too low, where (weak) thermal fluctuations would nevertheless overrun small spatial quenched fluctuations due to the interaction with random fields. In terms of the Ginsburg-Landau Hamiltonian, eq.(3), in the dimension $D = 3$ these two requirements impose the restrictions on the value of the reduced temperature parameter, $g^2 \ll |\tau| \ll h_0^2$, which automatically implies that the considered procedure of calculations is formally valid only at *finite* strength of random fields, $h_0 \gg g$. Under these restrictions both the "hand-waving" heuristic arguments (Section 2.2), and the formal replica calculations (Section 3) provide the results, eqs.(40) and (69), which nicely fit each other. Of course, the main thing here is not the actual value of the non-perturbative part of the free energy, but the physical mechanism, which provides it. According to the speculations of Section 2.2 the non-perturbative contributions appears due to rare large cluster spin flips (which in terms of replica calculations are described by the localised in space instanton configurations). It is crucial that such clusters (or the replica instantons) are supposed to be far from each other, so that they can be treated as non-interacting and independent. According to the obtained results, eqs.(40) and (69), the spatial density of these clusters, $\rho \sim \exp[-(const)|\tau|^{3/2}/(gh_0^2)]$, is exponentially small provided $|\tau| \gg (gh_0^2)^{2/3} \equiv \tau(h_0)$. In other words, when approaching the putative phase transition point from below (i.e. decreasing the value of $|\tau|$), at $|\tau| \sim \tau(h_0)$, the mean separation between clusters becomes comparable with their typical size. In this situation the whole scheme of calculations breaks down, as the state of the system becomes just a mess of locally ordered "up" and "down" regions, when it is impossible to separate the degrees of freedom connected with the flipped clusters from those of the ferromagnetic background. It is crucial that this happens at the temperature $\tau(h_0) \gg \tau_{GL} = g^2$ (well before the putative T_c of the expected ferromagnetic/paramagnetic phase transition) which is still far way from the Ginzburg-Landau crossover temperature τ_{GL} where the thermal fluctuations would become important, and therefore here the system can still be described at the mean-field level. Since at $|\tau| \sim \tau(h_0)$ the local order parameter (the absolute value of the local magnetisation) is still finite, it would be reasonable to expect that somewhere at these temperatures the system undergoes the phase transition of the *first order* into the disordered state. As for the nature of this disordered state, it should be stressed that (unless the replica symmetry is broken!) one should not expect one more phase transition from the spin-glass to the paramagnetic state. The point is that due to the presence of quenched random fields the quantity $\overline{\langle\phi\rangle^2}$ (which is the traditional replica-symmetric spin-glass order parameter) remains non-zero at all temperatures. Therefore one should expect not more than a crossover from "rather spin-glass" (just after the transition) to "rather paramagnetic" (at high temperatures) disordered states.

Unfortunately the analytic technique developed in this paper can not be directly applied for the description of this (rather exotic) disorder induced first-order phase transition. Nevertheless, since the situation seems to stay at the mean-field level, development of another analytic approach to this problem does not look completely hopeless.

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